

A Model Hamiltonian for Quantum Isolated Horizon

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Abstract

With a view to clarify some apparent inconsistencies in the classical and quantum theories of Isolated Horizon and motivated by the structure of the area operator in loop quantum gravity, a model Hamiltonian operator for the quantum Isolated Horizon is proposed. Known results of Isolated Horizon thermodynamics are used as inputs to fix the model. The proposal of the model is based on the facts that the Hamiltonian operator and the area operator associated with the quantum Isolated Horizon should have simultaneous eigenstates and in the correspondence limit one must obtain that the area of a classical Isolated Horizon is constant.

To study the thermodynamics of quantum black holes in the canonical ensemble using the Quantum Isolated Horizon(QIH) [1, 2] framework in quantum gravity, we need a well defined Hamiltonian operator for the QIH, since the canonical partition function explicitly involves the Hamiltonian. Usually, to study the thermodynamics of black holes in the canonical ensemble, the energy associated with the horizon is assumed to be some function of the area of the horizon. This is the outcome of our practice of viewing a theory primarily from a classical perspective. Here we take a complete upside down approach to the problem. This paper is all about the proposal of a model Hamiltonian operator for the QIH viewing the theory from a quantum perspective adapted from the method of the construction of area operator in loop quantum gravity(LQG)[3, 4] and considering Dirac's theorem of commuting observables [5]. Although the Hamiltonian is needed for the study of black hole thermodynamics from a purely quantum perspective, but the motivations originate from deeper underlying issues regarding the QIH framework [1, 2] and the theory of classical Isolated Horizon(IH)[6, 7, 8, 9, 10].

The phase space analysis of classical IH reveals that there is a CS theory on the IH [6]. CS theory being topological in nature, its Hamiltonian vanishes identically, which is a constraint and not a true Hamiltonian of the IH theory. It generates the gauge transformations on the IH. Looking from this classical viewpoint i.e. going to the quantum theory only through the classical one, the action of the corresponding operator is to annihilate the physical states of the CS theory. Hence, if the area operator associated with the QIH is a Dirac observable¹,

¹We mean *strong* Dirac observable only[11].

then it must not commute with the CS Hamiltonian operator. This in turn implies that the CS Hamiltonian operator and the area operator for the QIH *can not* have simultaneous eigenstates. Then a real problem arises : *What are the states on which the area operator for the QIH acts upon ?* All the papers in the literature related to QIH (or black hole) entropy [1, 2, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24](and elsewhere), *assume*² that the area operator acts on the quantum states of the QIH Hilbert space, which are annihilated by the CS Hamiltonian operator. For this assumption to be true the area and the CS Hamiltonian operators have to commute. Then the following question arises : *Is the area operator associated with the QIH a Dirac observable ?* Now, the obvious answer to this question is : *Yes*. That the area operator is a Dirac observable in LQG [3, 4], is quite a well known fact in literature. But, how shall we justify this known answer against the above discussed inconsistency?

There is yet another problem in the theory of classical IH. The covariant phase space analysis for a spacetime admitting IH as an inner boundary shows that there is a classical energy function associated with the IH satisfying a first law so as to have a Hamiltonian evolution of the phase space[8]. It is a firm indication towards the existence of a true classical Hamiltonian associated with the IH which will provide the notion of the energy associated with IH. But, this contradicts the fact that the CS Hamiltonian associated with the classical IH vanishes identically. This is a self-contradiction of the theory of classical IH.

So, one must wonder : *Where do the problems arise from ?*

A possible answer may be : *The way we look towards the theory is the source of these problems*. We always pass to the quantum theory only through the quantization of the classical one. But we never originally formulate the quantum theory and pass on to the classical one by studying the correspondence limit of the quantum theory³. To study the correspondence limit of a quantum theory is only ‘a consistency check’ for us and *not* a ‘method of derivation’ of the laws of classical physics. Adapting to this viewpoint in the context of the QIH theory, the above discussed problems actually have a solution, at least partial, if we approach the problems from a complete quantum perspective. The first law, which is an artifact of the *classical* theory of IH, is nothing but the proportionality of the variation of classical energy to that of the classical area, the proportionality constant being related to the surface gravity of the IH i.e. $\delta E_{IH}^t = (\kappa_{IH}^t/8\pi)\delta A_{IH}$ where t denotes the choice of time evolution vector field[8]. Wearing the ‘quantum spectacles’ one can view that on the right hand side of the first law, the classical area A_{IH} is the expectation value of the QIH area operator. Then the classical energy E_{IH} on the left hand side must also result from the expectation value of some Hamiltonian operator for the QIH. This clearly indicates that there must be a gauge invariant true Hamiltonian operator in the quantum theory which in the correspondence limit will give rise to this notion of local energy associated with the IH opposed to the CS Hamiltonian operator annihilating the physical states of the QIH. In fact, the clue to the solution is hidden in the structure of the area operator for the QIH which already is a self-adjoint and gauge invariant Dirac observable in the LQG framework. Here, we shall see that beginning from the structure of the area operator, we can at least

²Otherwise there is no meaning for counting the microstates of the QIH whose area eigenvalues lie within $\pm \mathcal{O}(\ell_p^2)$ of the classical area of the corresponding classical IH.

³Nature is the way it is. We are the ones who can see it classically only, which is the sole reason for this order of passage from classical to quantum. If human mind were sharp enough to formulate the quantum theory on the first hand, then we would have done so. Then, many fundamental problems which arose in classical physics and later solved in quantum theory, would not have risen at all.

predict the possible structure of the Hamiltonian operator associated with the QIH which is gauge-invariant, self-adjoint and commutes with the area operator. The commutativity ensures that the two operators have simultaneous eigenstates and the mean area is a constant of motion, leading to the conclusion that the classical IH has constant area.

The area operator is a gauge invariant, self-adjoint observable in loop quantum gravity(LQG) defined for any arbitrary two dimensional surface (S) embedded in the three dimensional spatial manifold (Σ) obtained from a specific foliation of the four dimensional spacetime manifold ($M \equiv R \otimes \Sigma$) by some preferred time evolution vector field (t)[3, 4]. The wave function of the spatial geometry is a functional of the spatial $SU(2)$ connection A . These functionals are equipped with some specific properties and are called *cylindrical functions*[25]. The spatial quantum geometry is denoted by a graph consisting of edges (links) and vertices (nodes). This ‘floating fish-net’ like structure is also known as spin network. Each edge is associated with a spin representation of the $SU(2)$ group. The wave function of the quantum geometry is written as

$$\Psi_{\Gamma,\psi}[A] = \psi(h_{\rho_1}(A), h_{\rho_2}(A), \dots, h_{\rho_n}(A)) \quad (1)$$

where $h_\rho(A)$ -s are the holonomies along the edges of the spin network, Γ is the collection of the ordered oriented paths and ψ is a smooth function on $[SU(2)]^n$ [4, 25]. The holonomy of the connection along a curve ρ embedded in Σ is given by

$$h(A, \rho) \equiv \mathcal{P} \exp \int_\rho A \quad (2)$$

which is a gauge invariant quantity. The momentum (E) conjugate to the connection variable (A) [25] acts on the wave function as operators given by the following functional derivative :

$$\frac{1}{8\pi\gamma G} \hat{E}_i^a(\tau, x) \Psi_{\Gamma,\psi}[A] = -i\hbar \frac{\delta \Psi_{\Gamma,\psi}[A]}{\delta A_a^i(\tau, x)} \quad (3)$$

The momentum (E) being a two form can be naturally integrated over a two surface. The corresponding operator can also be smeared on a two surface which leads to the flux operator in LQG which plays the key role in endowing the surface (S) with a quantum area [3, 4, 25, 26]. Since, from eq.(1), it is evident that $\Psi_{\Gamma,\psi}[A]$ consists of the holonomies along the edges of the spin network, the action of the momentum operator on the holonomy is the most crucial step, which is given by the functional derivative of the holonomy with respect to the connection as follows

$$\begin{aligned} \frac{\delta h(A, \rho)}{\delta A_a^i(\tau, x)} &= \int_0^1 ds \, \dot{\rho}^a(s) \, \delta^3(\rho(s), x) \\ &\quad \times [h(A, \rho_1) \tau_i h(A, \rho_2)] \end{aligned} \quad (4)$$

The quantity in the expression (4) is a two dimensional distribution and yields a well defined operator $\hat{E}_i(S)$ when smeared over a two dimensional surface (S), embedded in the three dimensional spatial manifold [3, 4, 26]. Geometrically, the operator $\hat{E}_i(S)$ signifies that when the path ρ intersects the surface S at a point Q (say), separating the path into two paths ρ_1 and ρ_2 , Q gets associated with a matrix $\pm i8\pi\gamma G \hbar \tau_i$ in the particular spin representation carried by the holonomy of the path ρ . The signature depends on the relative orientation

of the path (ρ) and the surface (S). Without any intersection the result is zero. The gauge invariant operator that can be constructed from $\hat{E}_i(S)$ and relevant for the construction of the area operator is simply $\sum_i \hat{E}_i^2(S)$. Its action on the holonomy of a path ρ carrying spin- j representation and intersecting S at a point is given by

$$\begin{aligned} & \sum_i \hat{E}_i^2(S) {}^{(j)}h(A, \rho) \\ = & {}^{(j)}h(A, \rho_1) \left[-(8\pi\gamma G\hbar)^2 \sum_i {}^{(j)}\tau_i^2 \right] {}^{(j)}h(A, \rho_2) \end{aligned}$$

Using $-\sum_i {}^{(j)}\tau_i^2 = j(j+1) \times \mathbf{I}$, the Casimir of the SU(2) group in the spin- j representation and the property ${}^{(j)}h(A, \rho) = {}^{(j)}h(A, \rho_1) \cdot {}^{(j)}h(A, \rho_2)$ for $\rho = \rho_1 \cup \rho_2$ on the right hand side of the above equation, one obtains an eigenvalue equation with the eigenvalue $(8\pi\gamma G\hbar)^2 j(j+1)$. The gauge-invariance of the operator $\sum_i \hat{E}_i^2(S)$ is manifested by the insertion of the matrix Casimir of the group and the real eigenvalue manifests the self-adjoint property, *which are all associated only with the intersection point*. This is the most crucial fact which will play the pivotal role in the phenomenology presented in this paper. Now, comparing with the classical definition of area of the surface S given by $\int_S dS \sqrt{n_a E_i^a n_b E_i^b}$ (n being normal to the surface S), it can be said that the path ρ upon intersecting the surface S provides it with a quantum of area $8\pi\gamma G\hbar \sqrt{j(j+1)}$.

Now, let there be N such intersections of the surface with the spin network edges. The edges are $\rho_1, \rho_2, \dots, \rho_N$ and the corresponding spin representations carried by the edges be j_1, j_2, \dots, j_N . But for multiple intersections on a surface matrices at different points get contracted spoiling the gauge invariance of the operator. To tackle this problem, the surface is partitioned into N pieces S_l , $l \in [1, N]$ such that $\cup_l S_l = S$. In the limit $N \rightarrow \infty$, the pieces get small enough to guarantee that there is a single intersection with each piece with a single edge. It should be remembered that the spin network is also dense enough so as to make large number of intersections with the surface S to make the above regularization⁴ scheme valid. The area operator associated with the surface S is given by

$$\hat{A}_S \equiv \lim_{N \rightarrow \infty} \sum_l \left[\sum_i \hat{E}_i^2(S_l) \right]^{1/2} \quad (5)$$

Hence the eigenvalue spectrum of the area operator will be given by

$$\hat{A}_S |\dots\rangle = 8\pi\gamma \ell_p^2 \sum_{l=1}^N \sqrt{j_l(j_l+1)} |\dots\rangle \quad (6)$$

where $|\dots\rangle$ denote a spin network state making intersections with the surface S and $\ell_p^2 = G\hbar$. j_l are half-integers⁵ The real eigenvalue spectrum tells us that the area operator is self-adjoint and the gauge-invariance has been guaranteed in the construction itself, specifically by the Casimirs appearing at the intersections. It is a Dirac observable [4]. The interesting property of the area operator that we are interested in is that *all the properties of the area operator*

⁴ See [4, 25] for detailed account on the regularization issue.

⁵The spectrum discussed here is only a part of the full area spectrum in LQG[4], which is generally considered to be relevant in the literature of quantum black holes and that is what we are interested in.

are actually carried by the individual intersection points where the edges of the spin network pierce the surface S .

The LQG area spectrum, in general, is unbounded above along with a vanishing lower bound for any arbitrary surface. But, the spectrum of the area operator belonging to a QIH Hilbert space has a positive lower bound along with a finite upper bound owing to the physical properties of the QIH. The Hilbert space of a QIH is that of a three dimensional $SU(2)$ Chern-Simons(CS) theory coupled to *punctures*(sources), carrying spin representations of the corresponding edges of the bulk spin network which pierce the IH (a null inner boundary of spacetime with the topology $R \otimes S^2$). The level of the source coupled CS theory is given by $k \equiv A_{cl}/4\pi\gamma\ell_p^2 \in I$, A_{cl} being the classical area of the IH. The physical states of the QIH belong to the singlet part of the Hilbert space which is given by $\mathcal{H}_S \equiv \text{Inv}(\otimes_{l=1}^N \mathcal{H}_l)$, where ‘Inv’ stands for gauge invariance on the QIH. The area spectrum of a QIH with N punctures and CS level k can be written as $8\pi\gamma\ell_p^2 \sum_{l=1}^N \sqrt{j_l(j_l+1)}$ with $1/2 \leq j_l \leq k/2 \forall l \in [1, N]$ [27]. Following the structure of \mathcal{H}_S , which is a gauge invariant direct product space of the Hilbert spaces associated with the *individual* punctures, we can further write \hat{A}_S as

$$\begin{aligned} \hat{A}_S \equiv & \hat{A}_{j_1} \otimes \hat{I}_{j_2} \otimes \cdots \otimes I_{j_N} + \hat{I}_{j_1} \otimes \hat{A}_{j_2} \otimes \cdots \otimes \hat{I}_{j_N} \\ & + \cdots + \hat{I}_{j_1} \otimes \hat{I}_{j_2} \otimes \cdots \otimes \hat{A}_{j_N} \end{aligned} \quad (7)$$

Even though, this above expression for the area operator usually does not appear in the literature, but it is a trivial structure to write down to ensure that \hat{A}_S acts on a quantum state of the QIH $|\phi_S\rangle \in \mathcal{H}_S \equiv \text{Inv}(\otimes_{l=1}^N \mathcal{H}_l)$, which can also be written as $|\phi_S\rangle \equiv \otimes_{l=1}^N |\phi_l\rangle$ following that the punctures are non-interacting and distinguishable[1, 2]. This particular way of writing the area operator is also motivated by the fact that the properties of the area operator of the QIH are actually carried by each *individual* punctures; recall that the Casimir of $SU(2)$ is inserted only at the puncture where an edge of the bulk spin network intersects the surface. Hence, the punctures are the building blocks of the QIH in the LQG framework and must play the roles of fundamentally important individuals while constructing an operator associated with the QIH.

Motivated by all the facts discussed up till now and with a view to approach the problem from a quantum viewpoint, it is prompting to write down the Hamiltonian operator for the QIH in the form of the area operator associated with the same. Hence, we write the QIH Hamiltonian as

$$\begin{aligned} \hat{H}_S \equiv & \hat{H}_{j_1} \otimes \hat{I}_{j_2} \otimes \cdots \otimes I_{j_N} + \hat{I}_{j_1} \otimes \hat{H}_{j_2} \otimes \cdots \otimes \hat{I}_{j_N} \\ & + \cdots + \hat{I}_{j_1} \otimes \hat{I}_{j_2} \otimes \cdots \otimes \hat{H}_{j_N} \end{aligned} \quad (8)$$

very similar to the area operator of the QIH given by eq.(7). Further, we propose that *any gauge invariant, self-adjoint operator associated with the QIH and which commutes with the area operator of the QIH, the contribution from a single puncture, carrying a spin- j representation, must be a polynomial of \hat{A}_j* . All these three properties are essential for a true Hamiltonian operator associated with the QIH, of which the reason for the requirement of commutativity is justified by two reasons. First of all, the area operator and the Hamiltonian associated with the QIH will have the simultaneous eigenstates, which are the states of the CS theory coupled to the punctures. Secondly, we must ensure that the expectation value of the area operator, which is equal to the classical area of the corresponding classical IH, must be a constant of motion, implying that the quantum theory properly leads to the most crucial property of the IH in the correspondence limit. So, we propose that the contribution

to the Hamiltonian from a single puncture of a QIH carrying a spin- j representation must be of the form

$$\hat{H}_j \equiv \sum_{n=0}^{\Lambda} p_n \hat{A}_j^n \quad (9)$$

where the coefficients (p -s) carry the burden of endowing the Hamiltonian operator with the correct dimensionality and Λ is a cut-off. Hence, the Hamiltonian operator associated with the QIH can be written as

$$\begin{aligned} \hat{H}_S \equiv \sum_{n=0}^{\Lambda} p_n \Big(& \hat{A}_{j_1}^n \otimes \hat{I}_{j_2} \otimes \cdots \otimes I_{j_N} + \hat{I}_{j_1} \otimes \hat{A}_{j_2}^n \otimes \\ & \cdots \otimes \hat{I}_{j_N} + \cdots + \hat{I}_{j_1} \otimes \hat{I}_{j_2} \otimes \cdots \otimes \hat{A}_{j_N}^n \Big) \end{aligned} \quad (10)$$

whose spectrum can be explicitly written as $\sum_{n=1}^{\Lambda} \sum_{l=1}^N p_n (8\pi\gamma\ell_p^2)^n [j_l(j_l+1)]^{n/2}$. Now, it is straightforward to see from expression (10) that $[\hat{H}_S, \hat{A}_S] \equiv \hat{0}$ which guarantees the fulfillment of the requirements that \hat{H}_S and \hat{A}_S have the simultaneous eigenstates which are that of the QIH Hilbert space and the expectation value of the area operator of a QIH is a constant of motion. If there is some evolution parameter τ which parametrizes the QIH then it is evident that $\frac{d}{d\tau} \langle \hat{A}_S \rangle = \frac{i}{\hbar} \langle [\hat{H}_S, \hat{A}_S] \rangle = 0$ i.e. in the correspondence limit the classical IH has *constant area* [6, 7, 8, 9, 10].

If the Hamiltonian would have resulted from the quantization of classical theory, then the p -s and Λ would have been automatically determined. Since we are aiming to explain the classical theory by studying the correspondence limit of the quantum theory, the only way of determining the p -s and Λ is to use our knowledge of the classical theory. One such method is to study the thermodynamics of QIH in the canonical ensemble and fix the p -s and Λ so as to produce the known results of the classical theory. Such an exhaustive thermodynamics analysis has been carried out in [29] which shows that the p -s and Λ only affect the expression of the intrinsic⁶ temperature of the horizon, which, for suitable choices of p -s, is given by $T = \frac{\ell_p}{8} \kappa \Lambda [1 + (1 - 2\pi^2\gamma^2)/\Lambda]$ (where the Boltzmann constant $k_B = 1$). The p -s can be explicitly written as $p_n = \frac{\kappa \ell_p}{(4\ell_p^2)^n \Gamma(n+2)}$ where κ is some unknown positive number. Arguing that this intrinsic temperature of the IH must be an *universal* one apart from carrying a scaling ambiguity signifying the infinite family of time evolution vector fields each associated with a first law on the IH [8], we choose $\kappa = [1 + (1 - 2\pi^2\gamma^2)/\Lambda]^{-1} \equiv \kappa(\Lambda)$. As a result, we obtain $T = \frac{\Lambda \ell_p}{8}$ where Λ is the only parameter left in the expression required to manifest ambiguity regarding the time evolution vector field. Hence, the complete structure of the model Hamiltonian operator for the QIH that we have proposed can be obtained by putting p_n and $\kappa(\Lambda)$ in the expression (10) and the spectrum of the Hamiltonian for an arbitrary spin sequence (j_1, \dots, j_N) on the QIH can be written as

$$\sum_{n=0}^{\Lambda} \sum_{l=1}^N \kappa(\Lambda) \ell_p \frac{(2\pi\gamma)^n}{\Gamma(n+2)} [j_l(j_l+1)]^{n/2} \quad (11)$$

⁶Not the one measured at infinity which is affected by the spacetime geometry away from the horizon e.g. differ by a red-shift factor for static spacetimes.

where $1/2 \leq j_l \leq k/2, \forall l \in [1, N]$. Furthermore, one can show [29] that the expectation values of the proposed Hamiltonian and area operators for the QIH are proportional. On taking the variation it yields a first law.

The situation of vanishing Hamiltonian does not appear in this current scenario and can be regarded as an artifact of our vision of approach from classical to quantum. It may be the case that there exists an explanation of the vanishing CS Hamiltonian from this quantum perspective. Even though we can not produce any quantitative argument in support of our viewpoint, but there is a qualitative one at our disposal which will conclude our discussion.

The phase space analysis of the IH [6] shows that the CS theory on the IH does not have a classical source coupling which would have made it a non-topological theory. Since the metric on the IH is degenerate such a source coupling could not have occurred as $\int *J \wedge A$ needs a nondegenerate metric to be well defined. Hence the absence of the source coupling and the topological nature of the CS theory saves the situation on the IH. Now, in [30], taking the Schwarzschild spacetime as an example, it has been explicitly shown that the equations of motion of the connection fields on the event horizon (a special case of IH) are that of a source coupled CS theory ! The procedure begins from considering the pullbacks of spacetime fields on the horizon and *using* Einstein equations of motion. But, in principle, it is impossible to derive those equations *directly* from the action principle simply because one will fail to write down the classical source coupled CS theory on the horizon due to the presence of a degenerate metric. Interestingly, here also a qualitative explanation exists from the view point presented in this work i.e. reaching the classical domain as a correspondence limit of the quantum theory. From our viewpoint, the equations of motion on the horizon must be thought of as the correspondence limit of the quantum equations which naturally admits the source coupling through the punctures at the horizon in the quantum geometric framework [1, 2]. The properties of the QIH are all inherited by those punctures only. We believe and predict that it is possible to show quantitatively that the classical equations of motion of the source coupled CS theory on the horizon can be and can only be consistently derived as a correspondence limit of the quantum equations involving wave functions of the quantum geometry, which will strengthen our round about approach to the scenario in the present case concerning the Hamiltonian.

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